

Adiabatic quantum dynamics of the Lipkin-Meshkov-Glick model

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We study the adiabatic dynamics of the Lipkin-Meshkov-Glick (LMG) model close to its quantum critical point by linearly switching the transverse field from an initial large value to zero. We concentrate our attention on the residual energy after the quench in order to characterize the level of diabaticity of the evolution. As a function of the characteristic time of the quench τ we identify three different regimes. For fast quenches the residual energy E_{res} is almost independent on τ . For slower quenches a second intermediate region appears in which a powerlike decay emerges with $E_{\text{res}} \sim \tau^{-3/2}$. By further slowing the quench rate, we find a third large- τ regime characterized by a different power law, $E_{\text{res}} \sim \tau^{-2}$. All these findings can be accounted for by means of an effective Landau-Zener approximation for the finite-size LMG model. We complete our description of the adiabatic dynamics of the LMG model through the analysis of the entanglement entropy of the evolved state.

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I. INTRODUCTION

Understanding the adiabatic quantum dynamics of many-body systems is central to many areas of physics and information science. In adiabatic quantum computation,¹ or in quantum annealing (see Ref. 2 for a review), the ultimate goal is to find the ground state of a complex system by adiabatically transforming the underlying Hamiltonian. Indeed any quantum algorithm can be efficiently implemented through the adiabatic evolution of a system from an initial exactly known state toward an unknown final one, which encodes the answer to the specific computational task.³ Once the time scale on which the Hamiltonian is varied is large compared to the typical inverse spectral gap of the system, the quantum adiabatic theorem⁴ ensures that if the system was prepared in the ground state of the initial Hamiltonian, at the end of the evolution, the quantum state will be the ground state of the final Hamiltonian. The bottleneck to the speed at which the algorithm is performed is thus given by those places where the instantaneous Hamiltonian has a spectrum where the gap closes in the thermodynamic limit, i.e., as the number of qubits increases. If the minimum gap closes faster than a power of the number of qubits, then the corresponding computational task is intractable.

The closing of a gap between the ground state and the first-excited level in the thermodynamic limit is a distinct feature of second-order quantum phase transition. It is responsible for the critical slowing down,⁵ and the evolution becomes necessarily not adiabatic. The problem of adiabatic dynamics close to a critical point, and the consequent defects formation, has originally been born in the study of phase transitions in the early universe.^{6,7} The recent extension to the quantum case^{8,9} has stimulated an intense theoretical activity.¹⁰⁻²⁹

In the search for a deeper understanding of the loss of adiabaticity on crossing a quantum critical point, an important role is played by exactly solvable models. The study of nonequilibrium many-body system is indeed a formidable task and the help of a tractable exactly solvable system can be of great help in testing approximate approaches, besides being of interest in itself. Most of the work done so far in this direction concentrated on one-dimensional quantum systems with short-range interaction. In this paper we would like to address a complementary limit, i.e., a model with infinite coordination (in the thermodynamic limit) but still amenable of an exact solution: the Lipkin-Meshkov-Glick model (LMG). First introduced by Lipkin *et al.*³⁰ in the context of nuclear physics, it was then adopted by the condensed-matter community as a paradigm of an infinitely coordinated solvable system.³¹ The result of a sudden quench in this model was recently discussed in Ref. 32. Here we present results in the opposite regime in which the system is dragged adiabatically through the critical point. As it will be shown in the following, although the phase transition is of mean-field nature, the dynamics leads to nontrivial results.

The paper is organized as follows. In Sec. II we introduce the Lipkin-Meshkov-Glick model and briefly review its properties, which are important for the purposes of this work. In the same section we also discuss how we solve numerically the dynamics (Sec. II A) and the observables used to quantify the departure from the adiabatic ground state (Secs. II B and II C). In this work we use the residual energy (the excess energy as compared to the adiabatic limit), the incomplete magnetization (the deficit magnetization as compared to the adiabatic limit), and the entanglement entropy. The numerical results together with the corresponding scaling ar-

guments are presented in Sec. III. In Sec. IV we present a critical assessment of our findings.

II. MODEL

The properties of the LMG model have been thoroughly scrutinized in the literature (see, e.g., Refs. 33–45, and references therein). Below we briefly recall few results that are relevant to the present paper. The LMG Hamiltonian describes a system of spins (1/2 in this work) interacting through an infinite-range exchange coupling and immersed in a transverse field. Assuming that the field is directed along the z direction, the Hamiltonian can be written as

$$H = -\frac{2}{N} \sum_{i < j} (S_i^x S_j^x + \gamma S_i^y S_j^y) - \Gamma \sum_i S_i^z, \quad (1)$$

where N is the number of the spins in the system, S_i are the Pauli operators, γ is the anisotropy parameter, and Γ is the transverse field. By introducing the total spin operator $\vec{S} = \sum_i \vec{S}_i$, the Hamiltonian can be rewritten, apart from an additive constant, as $H = -\frac{1}{N} [\mathcal{S}_x^2 + \gamma \mathcal{S}_y^2] - \Gamma \mathcal{S}_z$. The Hamiltonian hence commutes with \mathcal{S}^2 and does not couple states having a different parity of the number of spins pointing in the magnetic-field direction; that is, $[H, \mathcal{S}^2] = 0$ and $[H, \Pi_i S_i^z] = 0$. In the isotropic case, $\gamma = 1$, also the z component of \vec{S} is conserved, $[H, \mathcal{S}_z] = 0$.

In the thermodynamical limit the LMG model undergoes a second-order quantum phase transition at $\Gamma_c = 1$ characterized by mean-field critical exponents.³³ The magnetization in the x direction (or in the xy plane for $\gamma = 1$) vanishes when $\Gamma \rightarrow 1^-$ as

$$m = \begin{cases} (1 - \Gamma^2)^{1/2} & \Gamma \leq 1 \\ 0 & \Gamma > 1 \end{cases} \quad (2)$$

for all values of the anisotropy parameter γ . For $\Gamma > \Gamma_c$ and for any γ , the ground state is nondegenerate; while for $\Gamma < \Gamma_c$ it is doubly degenerate in the thermodynamical limit for any $\gamma \neq 1$, signaling the breaking of the Z_2 symmetry. The gap vanishes at the transition as

$$\Delta = [(\Gamma - 1)(\Gamma - \gamma)]^{1/2} \quad \text{for } \Gamma \geq 1. \quad (3)$$

For any finite N both the magnetization and the gap are modified (as any other physical observable). The finite-size scaling behavior is available in literature in all the relevant regimes (see, e.g., Refs. 33 and 40). The deviation from the thermodynamic limit for the gap $\delta\Delta_N = \Delta_N - \Delta$ and the magnetization $\delta m_N = m_N - m$ scale as

$$\begin{aligned} \delta\Delta_N &\sim N^{-1}, & \delta m_N &\sim N^{-1/2}, & \Gamma > 1, \\ \delta\Delta_N &\sim N^{-1/3}, & \delta m_N &\sim N^{-1/3}, & \Gamma = 1, \\ \Delta(N) &\sim e^{-aN}, & \delta m_N &\sim N^{-1}, & \Gamma < 1, \end{aligned} \quad (4)$$

for $\gamma > 1$ (where a is a constant) and

$$\delta\Delta_N \sim N^{-1}, \quad \delta m_N \sim N^{-1/2}, \quad \Gamma > 1,$$

$$\delta\Delta_N \sim N^{-1}, \quad \delta m_N \sim N^{-1/2}, \quad \Gamma = 1,$$

$$\delta\Delta_N \sim N^{-1}, \quad \delta m_N \sim N^{-1}, \quad \Gamma < 1, \quad (5)$$

for $\gamma = 1$, respectively. The scaling behavior of the gap is important in order to distinguish the various dynamical regimes in the adiabatic annealing. It is however important to stress at this point that the equilibrium gap is not necessarily the one responsible for the loss of adiabaticity. As we will see in Sec. III A, due to the parity conservation, the relevant gap for the dynamics is different from the equilibrium one (although with the same scaling behavior).

A. Adiabatic dynamics

The adiabatic dynamics is implemented by changing the external transverse field from an initial value $\Gamma \gg 1$ at $t_{\text{in}} = -\infty$, where the ground state of $H(t_{\text{in}})$ is completely dominated by the transverse field term with all the spins aligned along the $+\hat{z}$ direction, to $\Gamma = 0$, where the ground state is ordered in the xy plane. The annealing time is characterized by a time scale τ . More specifically we consider the case, as often in this type of problems, where the magnetic field is reduced linearly in time,

$$\Gamma(t) = -t/\tau \quad \text{for } t \in (-t_{\text{in}}, 0] \quad (6)$$

with $t_{\text{in}} \gg \tau$.

The problem we want to discuss is further simplified by the following observation. The initial state, the ground state of $H(t_{\text{in}})$, belongs to the sector of maximum spin $S = N/2$. Since S is a constant of motion, it is sufficient to restrict our attention to this subspace only. From now on we assume $S = N/2$ (for simplicity we consider N even). In the basis $|N/2, S^z\rangle$ ($S^z = -N/2, \dots, N/2$), the Schrödinger evolution of the state

$$|\psi(t)\rangle = \sum_{j=1}^{N/2+1} u_{2j-1}(t) |N/2, -N/2 - 2 + 2j\rangle \quad (7)$$

amounts to solving the following set of coupled equations:

$$i \frac{du_{2j-1}}{dt} = \sum_k A_{j,k} u_{2k-1}(t). \quad (8)$$

The odd amplitudes $|N/2, -N/2 - 1 + 2j\rangle$ do not couple because of parity conservation. In Eq. (8) A is a $(N/2+1) \times (N/2+1)$ symmetric matrix whose nonzero entries are given by

$$\begin{aligned} A_{j,j+1} &= -\frac{1}{4N} (1 - \gamma) a_{-N/2-2+2j} a_{-N/2+2j-1}, \\ A_{j,j} &= -\frac{1}{4N} (1 + \gamma) [a_{-N/2-3+2j}^2 + a_{-N/2-2+2j}^2] - \Gamma \left(-\frac{N}{2} - 2 + 2j \right) \\ &\quad + \frac{1}{4} (1 + \gamma), \end{aligned} \quad (9)$$

in terms of the usual angular-momentum raising operator matrix elements,

$$a_j = \left[\frac{N}{2} \left(\frac{N}{2} + 1 \right) - j(j+1) \right]^{1/2}. \quad (10)$$

Special values have the boundary terms of A given by

$$A_{1,1} = -\frac{1}{4N}(1+\gamma)a_{-N/2}^2 - \Gamma\left(-\frac{N}{2}\right) + \frac{1}{4}(1+\gamma),$$

$$A_{N/2+1,N/2+1} = -\frac{1}{4N}(1+\gamma)a_{N/2-1}^2 - \Gamma\left(\frac{N}{2}\right) + \frac{1}{4}(1+\gamma). \quad (11)$$

Equation (8) was integrated via standard numerical methods with initial conditions given by the amplitudes of the ground state of $H(t=t_{\text{in}})$.

B. Residual energy and incomplete magnetization

A natural way of quantifying the degree of adiabaticity of the evolution is to measure the residual energy defined as

$$E_{\text{res}} = E_{\text{fin}} - E_{\text{gs}}, \quad (12)$$

where E_{gs} is the ground-state energy of $H(t_{\text{fin}})$ and $E_{\text{fin}} = \langle \psi(t_{\text{fin}}) | H(t_{\text{fin}}) | \psi(t_{\text{fin}}) \rangle$ is the average energy of the final time-evolved state $|\psi(t_{\text{fin}})\rangle$. Obviously E_{fin} , and hence E_{res} , depends on the annealing time τ , the slower the evolution the closer the final energy to E_{gs} and hence the smaller the residual energy.

An alternative way of quantifying the degree of adiabaticity of the evolution is in terms of the incomplete magnetization in the final state defined by

$$m_{\text{inc}} = m_{\text{gs}} - m(t), \quad (13)$$

where m_{gs} is the static magnetization of the ground state of $\Gamma=0$ and m_{gs} is the average magnetization of the final evolved state. Following Botet *et al.*,³¹ the magnetization m has been defined through

$$m^2 = \frac{4}{N^2} \langle \psi | \mathcal{S}_x^2 + \delta_{\gamma,1} \mathcal{S}_y^2 | \psi \rangle, \quad (14)$$

where the expectation value can be taken either the ground state, for m_{gs} , or in the evolved state, for $m(t)$. As discussed by Botet *et al.*,³¹ the previous definition differs from that of the spontaneous magnetization; however, it is more amenable for finite-size systems and it reduces to the spontaneous magnetization in the thermodynamic limit.

Since we are dealing with a model where the coupling has an infinite range, the incomplete magnetization is an appropriate way to characterize the loss of adiabaticity. In this case a correlation length characterizing the typical distance between defects, along the lines followed for short-range models, cannot be introduced.

In the Ising limit, $\gamma=0$ at $\Gamma(t=0)=0$, the residual energy and the incomplete magnetization are related as they both depend only on the average value $\langle \psi(t=0) | \mathcal{S}_x^2 | \psi(t=0) \rangle$. The residual energy per site can be expressed as

$$\frac{E_{\text{res}}}{N} = -\frac{1}{N^2} \langle \psi(t=0) | \mathcal{S}_x^2 | \psi(t=0) \rangle + \frac{1}{4}. \quad (15)$$

The incomplete magnetization is given by

$$m_{\text{inc}} = 1 - \sqrt{\frac{4}{N^2} \langle \psi(t=0) | \mathcal{S}_x^2 | \psi(t=0) \rangle}. \quad (16)$$

C. Entanglement entropy

In addition to the previous observables, it was recently shown that important information of the lack of adiabaticity in the system can be acquired by analyzing the entanglement entropy S .^{12,15} The study of entropy and other measures of entanglement has been recently intensively studied to characterize both equilibrium and nonequilibrium quantum many-body systems (see Ref. 46 for a review). In the case of the LMG model the ground-state entanglement entropy was studied in Refs. 47–51. In the present work we study the time evolution of S during an adiabatic evolution.

Given a bipartition of the system in L and $N-L$ spins, the entanglement entropy associated to the reduced density matrix of one of the subsystems, say ρ_L , is defined as

$$S_L = -\text{Tr}(\rho_L \log_2 \rho_L). \quad (17)$$

The entropy S_L measures the entanglement between the L spins and the rest of the system.

The entanglement entropy is straightforwardly evaluated by noticing that, being the states $|\mathcal{S}=N/2, \mathcal{S}_z\rangle$ symmetric under any permutations of the sites and being the maximum value of the total spin achievable only with the maximum value of the spin in each subsystem, the following decomposition holds:⁴⁸

$$|N/2, \mathcal{S}_z\rangle = \sum_{l=0}^L p_l^{1/2} |L/2, l-L/2\rangle \otimes |(N-L)/2, n-l-(N-L)/2\rangle.$$

In the previous decomposition n and l indicate, respectively, the number of up spins in the system and in the partition which defines the L sites. The coefficients appearing are defined as $p_l = L!(N-L)!n!(N-n)!/[l!(L-l)!(n-l)!(N-L-n+l)!N!]$. With the knowledge of the representation of the evolved state in the basis $|N/2, \mathcal{S}_z\rangle$ and by using previous decomposition, it is immediate to trace out the $N-L$ spins to obtain the reduced density matrix ρ_L and calculate its entropy.

III. RESULTS

The results presented below were obtained by integrating numerically Eq. (8). We verified that, as for the initial time of the evolution, it is enough to consider $t_{\text{in}}=-5\tau$ for faster sweeps ($1 < \tau < 500$) and $t_{\text{in}}=-2\tau$ for slower ones. We checked (data not reported) that our results do not depend on the precise value of t_{in} . We considered systems up to $N=1024$ spins and annealing times up to $\tau \sim 10^3-10^4$.

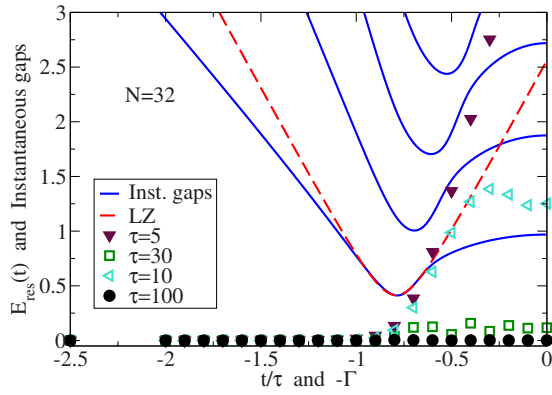


FIG. 1. (Color online) Residual energy $E_{\text{res}}(t)$ versus t for a given instance with $N=32$ and $\gamma=0$ of the LMG model at different values of τ . The solid lines are the lowest-lying instantaneous spectral gaps as a function of Γ . The red-dashed line is the best fit to the lowest gap used to calculate the Landau-Zener transition rates.

A. Residual energy and incomplete magnetization data

In order to understand the mechanism that leads to breakdown of adiabaticity in the LMG model, it is instructive to start with one particular example. In Fig. 1 we chose a system with $N=32$ spins and $\gamma=0$, showing the time evolution of the residual energy for different values of the annealing time τ . We also plot the instantaneous *accessible* gaps (thick solid lines) obtained by diagonalizing the Hamiltonian at any given Γ . As one can see, as soon as the system loses the adiabaticity, for fast annealing, it starts to ramp up in energy. The characteristic time scale for breaking of adiabaticity is however not given by the equilibrium smallest gap. As noticed in Sec. II, the dynamics is restricted to the subspace with fixed total spin $S=N/2$ and can involve only states with the same parity of S^z .⁵² Hence, the first gap relevant for the dynamics, that we call *dynamical* gap, is the energy difference between the ground state and the *second*-excited state, the smallest gap being forbidden by parity conservation of the total spin along the z axis. As shown in Fig. 2, the dynamical gap exhibits the same critical behavior of the excitation gap.³³ Both close polynomially in the thermodynamical limit with the same dynamical exponent $z=1/3$ in $\Delta_c \sim N^{-z}$. This is usually accompanied by a polynomial-like de-

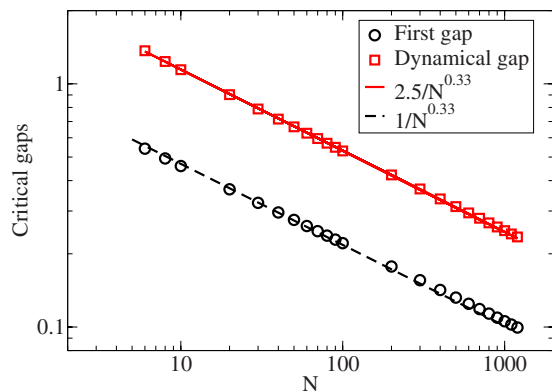


FIG. 2. (Color online) Smallest gap and dynamical gap at the critical point as functions of the size of the system.

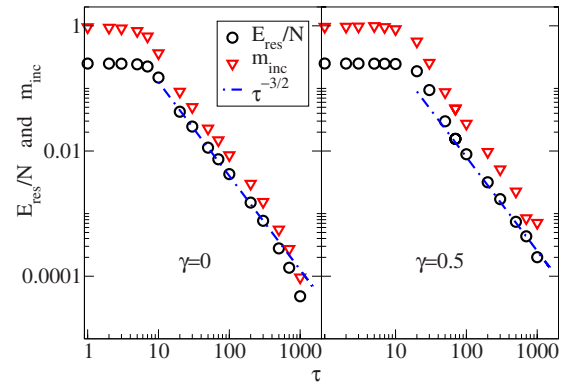


FIG. 3. (Color online) Residual energy per site and incomplete magnetization for the LMG model with $N=1024$ for different values of the anisotropy parameter γ . In all cases, for slow enough quenches, a power-law behavior $\tau^{-3/2}$ appears.

crease of the residual energy with increasing annealing time τ . This is indeed the case, for both the residual energy and the incomplete magnetization, as shown in Fig. 3 and, more in detail for $\gamma=0$, in Fig. 4. The behavior appears to be qualitatively independent on the value of the anisotropy parameter γ for $\gamma < 1$ (see Fig. 3); this was expected due to the fact that the minimum gap has the same large- N behavior irrespective of the anisotropy. In the following only the case $\gamma=0$ will be discussed.

Inspection of Fig. 4 reveals three different regimes. For fast quenches the dynamics involves almost all the levels (see, e.g., Fig. 1 for $\tau=5$). The residual energy per site is near to its maximum and shows very little dependence on the size of the system and on the annealing time τ . For larger values of τ , a second intermediate region appears in which a powerlike decay emerges, with $E_{\text{res}} \sim \tau^{-3/2}$. Finally, by further slowing the quench rate, a third large- τ regime characterized by a different power law, $E_{\text{res}} \sim \tau^{-2}$, emerges. We briefly discuss the emergence of the last two regimes by means of a Landau-Zener (LZ) approach adapted to the present problem.

The argument follows closely the one given in Ref. 8. The probability of exciting the system into the first-excited state, obtained by the Landau-Zener formula

$$P_{LZ} \simeq e^{-\alpha \Delta^2 \tau} \tag{18}$$

with $\alpha = \pi/4$, gives a lower bound to the true transition probability as it ignores the transitions to all the other excited

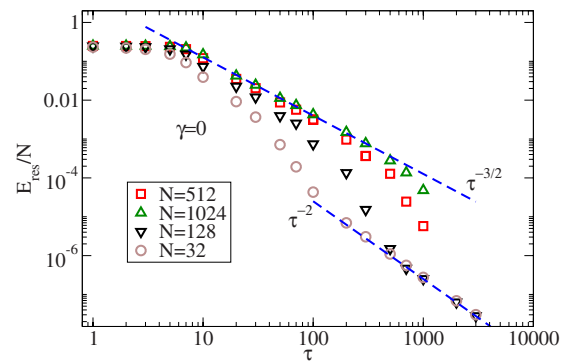


FIG. 4. (Color online) Residual energy per spin as a function of τ for $\gamma=0$ compared with different power-law behaviors.

levels. Using the scaling of the critical-point gap with the number of spins, $\Delta \sim N^{-1/3}$, it is possible to determine maximum system size for a defect-free quench once the probability for this to occur is fixed to the value \tilde{P}_{ex} . This gives

$$\frac{1}{N_{\text{free}}} \sim \left(\frac{|\ln \tilde{P}_{\text{ex}}|}{\alpha} \right)^{3/2} \frac{1}{\tau^{3/2}}. \quad (19)$$

One can consider $1/N_{\text{free}}$ as an estimate of the fraction of the flipped spins after the quench. The residual energy per site in the LMG model can then be evaluated to be

$$\frac{E_{\text{res}}}{N} \sim \frac{1}{N^2} \frac{N}{N_{\text{free}}} N \sim \frac{\text{const}}{\tau^{3/2}}. \quad (20)$$

This simple estimate is in good agreement with the numerical data in the intermediate regime of Fig. 4.

For short-range models the same power law of Eq. (20) can be also derived by determining the spatial scale over which defects occur.⁸ We tried to apply the arguments of Zurek *et al.*⁸ to the LMG model by identifying the correlation length with the coherence number introduced in Ref. 31. The procedure we followed, however, does not lead to the correct exponent. We have reasonable confidence that the failure in obtaining the correct scaling with this second method may be related to the above identification and the consequent definition of defect density. It would be interesting to find the correct argument in order to extend the approach by Zurek *et al.*⁸ or Polkovnikov⁹ to infinite range models.

B. Effective two-level approximation

As already mentioned before, there is, for slower quenches, a further crossover to a different power law. Can one explain also this behavior by using a Landau-Zener argument? To this end, it is important to refine this comparison and to understand to which extent the dynamics of a many-body system described by the LMG model can be described by two (many-body) levels. In general, in a many-body system there will be a number of avoided crossings and multiple LZ transitions including interference between them. Only when a single avoided crossing is dominant and well separated from the others a two-level approximation is appropriate. A detailed analysis of this issue is summarized in Figs. 5 and 6 where we show the case of $N=32$ as an example. Our analysis starts by extracting the best dynamical minimum gap and adapting to it the following two-level Hamiltonian:

$$H_{\text{LZ}} = \begin{pmatrix} -\Omega_{\text{LZ}}(\Gamma - \Gamma_0) & \Delta_{\text{LZ}} \\ \Delta_{\text{LZ}} & \Omega_{\text{LZ}}(\Gamma - \Gamma_0) \end{pmatrix}. \quad (21)$$

In the effective Landau-Zener problem Ω_{LZ} , Δ_{LZ} , and Γ_0 are the fitting parameters and $\Gamma = -t/\tau$. In Fig. 1 the dashed line represents the instantaneous gap of the Hamiltonian (21) suited to the case of $N=32$. From here we compare the results of the full LMG model with those obtained using LZ theory. As shown in Fig. 5, the excitation probability in the LMG model for slow enough quenches coincides with that of the effective LZ problem. It appears that this approximation

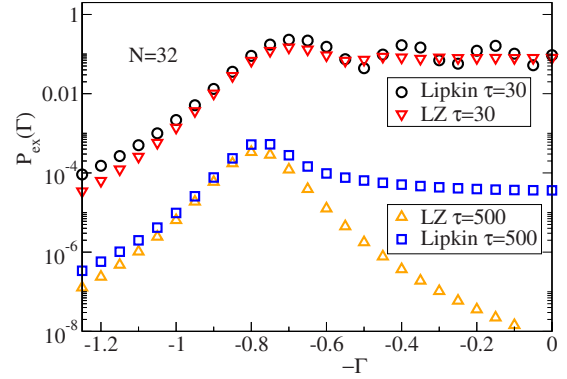


FIG. 5. (Color online) Comparison between the excitation probabilities as functions of Γ of the LMG model with $N=32$ and of its effective LZ approximation for different values of τ .

is good also in the estimate of the asymptotic value of the probability for $10 < \tau < 100$. Deviations come predominantly from the more enhanced oscillations of the post crossing region in the LMG model. For large τ 's the asymptotic value obtained from the effective two-level system gives a very poor approximation to the actual data. This can be traced back to the presence of further crossings which are obviously neglected in the two-level approximation. In our LZ scheme this can be effectively corrected by approximating the LZ crossing probability to the time before the next level crossing comes into play. This is explained below.

As found by Vitanov,⁵³ it is possible to define the duration of a single LZ as the time required by the probability for jumping from zero to its asymptotic value linearly and with the slope calculated at the crossing point. Using Γ as time scale one can write

$$\Gamma_{\text{jump}} \sim \frac{P(\infty)}{P'(\Gamma_{\text{cross}})}. \quad (22)$$

This time turns out to be *exponentially* divergent with τ for large τ .⁵³ This means that, for slow quenches, consecutive

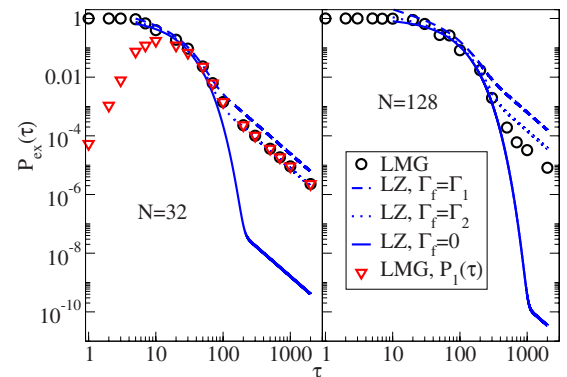


FIG. 6. (Color online) Excitation probability as a function of τ for LMG systems (circles) of two different sizes compared to that of one of the respective LZ-effective models (line) for different final times. For the LZ models the first two terms of the Vitanov approximation have been used. For the case of LMG with $N=32$, the probability of exciting the first level is also presented (triangles).

LZ transitions are not independent. In a first crude approximation, we can guess that the consequence of this is simply to stop the probability from relaxing toward the asymptotic value when the system has reached the second crossing. The presence of a power-law regime $\sim \tau^{-2}$ for extremely slow dynamics is a clear consequence of the finite duration of the evolution. In the original works by Landau and by Zener, the final time is supposed to be $t_f = \infty$; here the evolution is stopped at $\Gamma_f = -t_f/\tau = 0$ for the LMG model and at $t_{(LZ)f} = -\Gamma_{(LZ)f}\tau$ for the effective LZ with $\Gamma_{(LZ)f} = \Gamma_f - \Gamma_0$. An accurate analysis of the finite-time Landau-Zener model (FTLZ) has been done in Ref. 53, where it is shown that the transition probability reads, in this case,⁵⁴

$$P_{(\text{FTLZ})}(\tau) \sim P_{\text{LZ}}(\tau) + \frac{[1 - 2P_{\text{LZ}}(\tau)]}{16\Delta_{\text{LZ}}^4 \frac{\tau^2}{\Omega_{\text{LZ}}^2} \left(1 + \frac{\Omega_{\text{LZ}}^2}{\Delta_{\text{LZ}}^2} \Gamma_{(LZ)f}^2\right)^3} \quad (23)$$

with $P_{\text{LZ}}(\tau) = e^{-\pi\Delta_{\text{LZ}}^2\tau/\Omega_{\text{LZ}}}$. As it can be immediately seen from Eq. (23), by sending the final time to infinity, the usual LZ probability is recovered. The crossover rate $\hat{\tau}$ to the τ^{-2} scaling is obtained by equating the two terms on the right-hand side of Eq. (23). In the limit $\frac{8}{\pi} \left(1 + \frac{\Omega_{\text{LZ}}^2}{\Delta_{\text{LZ}}^2} \Gamma_{(LZ)f}^2\right)^{3/2} \gg 1$ the crossover time is approximated by

$$\hat{\tau} \sim \frac{\Omega_{\text{LZ}}}{4\Delta_{\text{LZ}}^2} \frac{1}{\left(1 + \frac{\Omega_{\text{LZ}}^2}{\Delta_{\text{LZ}}^2} \Gamma_{(LZ)f}^2\right)^{3/2}}. \quad (24)$$

In Fig. 6 we compared the excitation probabilities of LMG systems of different sizes with their single-LZ approximations. The probabilities for the effective models are evaluated for three different final time: $\Gamma_f = 0, \Gamma_1, \Gamma_2$, where the last two are the positions, respectively, of the minimum gap between the ground state and the second-excited level and the minimum gap between the first- and the second-excited levels. As it can be seen, the agreement is quite good and one can reproduce in this way also the regime with the τ^{-2} behavior.

C. Entanglement entropy

We finally would like to discuss the behavior of the entanglement entropy, which was already used as a tool to characterize adiabatic many-body dynamics in Refs. 12 and 15. Our results are summarized in Fig. 7. In the left lower panel the entanglement entropy, for a block of size $L = N/2$, of the state evolved down to $\Gamma_f = 0$ is plotted as a function of the quench time τ . For fast quenches, $\tau \rightarrow 0$, the state does not evolve (it remains in a nearly factorized state); thus, the entanglement necessarily tends toward zero. For very slow dynamics $\tau \rightarrow \infty$, since we are dealing with finite systems, the evolution eventually becomes adiabatic and the entanglement picks up the value it assumes in the final ground state $S_{\text{gs}}(\Gamma_f = 0) = 1$ independently on the subsystem size.⁴⁸ Between this two limiting behaviors, the entropy reaches a size-dependent maximum at an intermediate value of τ . An interesting feature is that the presence of a finite minimum gap can be easily connected with a time scale for the decaying of

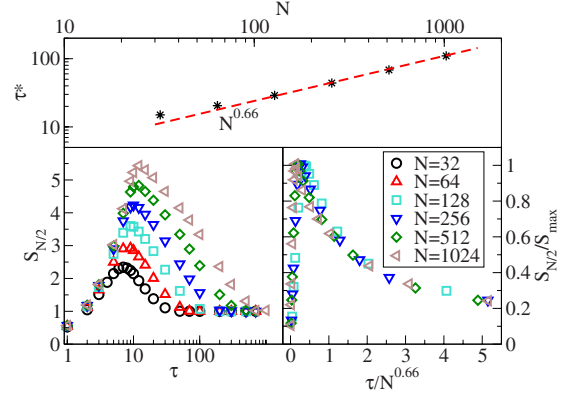


FIG. 7. (Color online) Left lower panel: Entanglement entropy of a block of $L = N/2$ spins as a function of the quench time τ . Right lower panel: entanglement entropy of a block of $L = N/2$ divided by its maximum value as function of the rescaled variable $\tau/N^{0.66}$. Upper panel: The time scale τ^* , see the text for the definition, as a function of system size N .

the entanglement. A possible choice for this time scale consists in selecting the τ^* , at which the entropy has reduced by half the value of its peak with respect to the slow quench limit

$$S_{N/2}(\tau^*) = \frac{(S_{\text{max}} - 1)}{2} + 1. \quad (25)$$

In the upper panel of Fig. 7, τ^* determined in this way is shown as a function of the system size N . For large N , a powerlike behavior emerges with an exponent of ~ 0.66 hinting at a relation

$$\tau^* \sim \frac{1}{\Delta^2}. \quad (26)$$

In the lower right panel of Fig. 7, the entanglement $S_{N/2}$ divided by its maximum value S_{max} is plotted as a function of the rescaled variable $\tau/N^{0.66}$, showing, for large systems ($N \geq 128$), a collapse of all data on the same curve. Note that Eq. (26) expresses exactly the same energy-time relation found in the usual LZ system [see Eq. (18)], so that the correspondence stated in Secs. III A and III B is again supported.

IV. CONCLUSIONS

In this paper we have studied the adiabatic quantum dynamics of the LMG model in a transverse field across its quantum critical point. We focused our attention on the residual energy after the quench, analyzing its behavior as a function of the annealing time, in order to evaluate the extent of diabaticity of the evolution. The dynamics is restricted to a subspace of definite total spin and parity of its projection along the z axis due to the symmetries of the Hamiltonian. The results appeared to be qualitatively independent of the value of the XY -anisotropy parameter γ , except for the fully isotropic XX case at $\gamma = 1$, where the further conservation of S^z plays an important role. By starting the evolution in the

ground state for very large values of the transverse field Γ , and then reducing $\Gamma(t)$ linearly to zero, three regimes in the residual energy are identifiable. The first one, corresponding to fast quenches, is strongly not adiabatic, involves transitions from the ground state toward many excited states, and is characterized by a residual energy near to its saturation value. In the intermediate regime, the lowest critical dynamically accessible gap starts dominating the evolution, inducing a residual energy per site that decays in a powerlike manner as $\tau^{-3/2}$. The third large- τ region, where the residual energy decays as τ^{-2} , is understood by taking into account the presence of additional level crossings. In the effective Landau-Zener description we used in this paper, this results in the requirement of considering a finite-time Landau-Zener

sweep. As shown by Vitanov a finite-time sweep leads to a polynomial (in τ) contribution to the LZ transition probability, which is dominant for very slow sweeping rates. Notice that this τ^{-2} regime, usually described as the general deviation from adiabaticity derived from the adiabatic theorem for very slow evolutions,^{55,56} emerges here in an alternative way through the parallelism with an effective FTLZ model.

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